

Compact null hypersurfaces and collapsing Riemannian manifolds

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Abstract

Restrictions are obtained on the topology of a compact divergence-free null hypersurface in a four-dimensional Lorentzian manifold whose Ricci tensor is zero or satisfies some weaker conditions. This is done by showing that each null hypersurface of this type can be used to construct a family of three-dimensional Riemannian metrics which collapses with bounded curvature and applying known results on the topology of manifolds which collapse. The result is then applied to general relativity, where it implies a restriction on the topology of smooth compact Cauchy horizons in spacetimes with various types of reasonable matter content.

1. Introduction

The concept of collapsing Riemannian manifolds introduced by Cheeger and Gromov [3, 4] has been the subject of a considerable amount of work in Riemannian geometry in recent years. A sequence of Riemannian manifolds is said to collapse if the injectivity radius tends uniformly to zero while the sectional curvature remains bounded. A manifold is said to collapse if it admits a collapsing sequence of Riemannian metrics. If this sequence can be chosen so that the diameter of the Riemannian metrics remains bounded then the manifold is said to collapse with bounded diameter. In this paper results on collapsing Riemannian manifolds are applied to a question in Lorentzian geometry which is of interest in general relativity. A Lorentzian manifold (M, g) is defined to be a four-dimensional manifold M with a metric tensor g of signature $(-, +, +, +)$. Hyperplanes in the tangent space to M at some point p are of three types; a hyperplane is said to be spacelike, null or timelike if the pull-back of the metric $g(p)$ to this hyperplane is positive definite, degenerate or indefinite, respectively. A hypersurface is said to be spacelike, null or timelike if its tangent space at each point is of the corresponding type. If H is a null hypersurface in M and if H can be approximated by spacelike hypersurfaces in some appropriate sense (the precise sense which is of relevance in the following will be specified later) then a one-parameter family of Riemannian metrics is obtained which degenerates as the parameter tends to some limiting value. However this is not enough to say that this family of metrics collapses; it is also necessary to know that the curvature of these metrics is uniformly bounded. In the following a situation will be exhibited where this condition is satisfied.

Let H be a smooth null hypersurface in a Lorentzian manifold. If p is a point of H

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there is precisely one direction in the tangent space to H at p such that if a vector L spans this direction then $g(L, X) = 0$ for all vectors X tangent to H at p . In fact this is just the normal direction to H . The curves in H which are tangent to this preferred direction are called the generators of H . They are geodesics (see e.g. [20], p. 65). Locally on H it is possible to choose a smooth vector field L which on each generator coincides with the tangent vector to an affinely parametrized geodesic. This vector L is only defined up to a constant multiplicative factor on each generator. Let \tilde{L} be a smooth extension of L to a null vector defined on an open neighbourhood in M of the domain of L and define θ to be the restriction of $\text{div} \tilde{L}$ to H . The quantity θ is called the expansion of L and does not depend on the extension \tilde{L} chosen. If L is scaled by a factor which is constant on each generator then θ is scaled by the same factor. A null hypersurface is called divergence-free if $\theta = 0$ for some (and hence any) choice of L in a neighbourhood of each point of H .

The main result of this paper is that in the presence of certain restrictions on the Ricci tensor r of the metric g , the topology of a divergence-free compact null hypersurface in a Lorentzian manifold is strongly constrained.

Theorem 1 Let (M, g) be a Lorentzian manifold whose Ricci tensor is zero and let H be a divergence-free compact smooth null hypersurface in M . Then the manifold H collapses with bounded diameter.

This is a special case of the following more general result.

Theorem 2 Let (M, g) be a Lorentzian manifold and let H be a divergence-free compact smooth null hypersurface in M . Suppose that $r(Z, X) = 0$ when Z is normal to H and X is tangent to H . Then the manifold H collapses with bounded diameter.

To understand the significance of the conclusion it is necessary to know how much the topology of a three-dimensional manifold is constrained by the condition that it collapses. An introductory account of the topological restrictions implied by collapsing can be found in [14]. An example of a class of compact three-dimensional manifolds which do not collapse are those which are hyperbolic, i.e. which admit a metric of constant negative curvature. It turns out that collapse with bounded diameter is a much stronger restriction than just collapse. A compact three-dimensional manifold which collapses with bounded diameter must either be a Seifert manifold or admit a geometric structure of type Sol. Recall that a Seifert manifold is one which admits a foliation by circles [18]. Any such manifold admits a geometric structure in the sense of Thurston. More precisely, the class of Seifert manifolds coincides with the class of manifolds which admit a geometric structure corresponding to one of six of the eight Thurston geometries. The two remaining geometries are Sol, which was mentioned above, and hyperbolic geometry, which is not compatible with collapsing. Thus a compact three-dimensional manifold which collapses with bounded diameter admits a geometric structure which is not hyperbolic. All other geometric structures can occur, as discussed in more detail in section 3.

The argument which leads to the statement on the existence of a geometric structure will now be sketched. (Cf. [16], [19].) This uses the notion of Gromov-Hausdorff convergence of metric spaces (see e.g. [14]). The metric spaces of relevance here are obtained by considering the distance function associated to a Riemannian metric. The Gromov precompactness theorem [10] says that a sequence of Riemannian metrics with bounded

curvature and diameter has a subsequence which converges in the Gromov-Hausdorff sense to some metric space. A theorem of Fukaya ([7], Theorem 0.6) then shows that the limiting space is isometric to the quotient of a Riemannian manifold by an isometric action of $O(n)$. This means in particular that the limiting space will have an open dense subset which is a Riemannian manifold. This open set corresponds to the principal orbit type of the $O(n)$ action [13]. For the present purposes the dimension of the limiting space will be defined to be the dimension of this open dense subset. There are now three possibilities: the dimension of the limiting space may be zero, one or two. If the dimension is zero then H is by definition an almost flat manifold [9, 1, 17]. Any almost flat three-dimensional manifold either admits a flat metric or a geometric structure of type Nil. If the limiting space is of dimension two then by a theorem of Fukaya ([6], Proposition 11.5) it is an orbifold. Then another result of Fukaya (the fibre bundle theorem) [8] shows that H is a Seifert manifold. Finally, if the limiting space has dimension one, it is homeomorphic either to a circle or a closed interval. In particular it is an orbifold. A result of Tuschmann [19] then implies that it must be flat or admit a geometric structure of type Nil or Sol.

Theorems 1 and 2 are proved in section 2. The idea is to approximate the null hypersurface H by spacelike hypersurfaces in an appropriate sense and show that the family of Riemannian metrics which arises in this way has bounded curvature. This can then be used to show that H collapses. In section 3 the applications of the theorem to general relativity are described. In general relativity there is an important class of compact null hypersurfaces, the compact Cauchy horizons, which are divergence-free. The hypotheses on the curvature in Theorems 1 and 2 follow from certain hypotheses on the matter content of spacetime. In particular, Theorem 1 corresponds to the case where spacetime is empty.

2. Boundedness of the curvature

Let H be a smooth compact null hypersurface in a Lorentzian manifold (M, g) . There exists a one-dimensional distribution (field of directions) on M which is timelike in the sense that it can be spanned locally by a vector T with $g(T, T) < 0$. Let W be an open neighbourhood of H with compact closure. If g is time orientable then there exists a global smooth vector field T on W spanning the given distribution. It may be assumed without loss of generality that $g(T, T) = -1$. In this case a family of tensors is defined by:

$$g_\lambda(X, Y) = g(X, Y) + \lambda g(T, X)g(T, Y) \quad (1)$$

For λ sufficiently close to zero g_λ is Lorentz metric. Even if g is not time orientable it is still possible to do a similar construction. The choice of the unit vector T at a given point is unique up to a sign and the definition of g_λ does not change when T is replaced by $-T$. Hence if g_λ is defined by (1) using local vector fields T , these locally defined tensors fit together to define a smooth object g_λ globally on W . If X is a non-zero vector tangent to H then $g(X, X) \geq 0$. It follows that $g_\lambda(X, X) > 0$ for $\lambda > 0$. Thus H is spacelike with respect to the Lorentz metrics g_λ for each $\lambda > 0$. Let h_λ be the pull-back of g_λ to H . Then h_λ is a Riemannian metric for $\lambda > 0$ and is degenerate for $\lambda = 0$. The diameter of the metric h_λ is bounded as $\lambda \rightarrow 0$ while its volume tends to zero.

It will now be shown that the curvature of h_λ remains bounded as $\lambda \rightarrow 0$. The curvature will be computed in a local frame adapted to H . This frame will only be defined at points of H . Let Z be a vector field defined on a neighbourhood of a point of H which is non-vanishing and tangent to the generators of H . Consider a unit timelike vector field T as above. Let Π denote the two-dimensional distribution which is the intersection of the tangent space of H with the normal to T with respect to g_λ . It follows from the definition of g_λ that Π does not depend on λ and that the induced metric on Π is also independent of λ . Let X and Y be an orthonormal basis of Π . Let $z = g_\lambda(Z, Z)$. This is positive for $\lambda > 0$ and tends to zero as $\lambda \rightarrow 0$. For $\lambda > 0$ let $\hat{Z} = z^{-1/2}Z$ and let \hat{U} be the unit normal vector to H . It can be rescaled to give a vector U which extends smoothly to $\lambda = 0$. The vector U is proportional to Z for $\lambda = 0$ and it can be assumed without loss of generality that $U = Z$ there. Note that for $\lambda = 0$:

$$dz/d\lambda = (g(T, Z))^2 > 0$$

It follows that $U - Z = zD$ for some smooth vector D . Let $u = -g_\lambda(U, U)$. Then $u = z - z^2 g_\lambda(D, D)$. In particular, it follows that u/z has a smooth extension to $\lambda = 0$ and takes the value one there. Note that since the generators are geodesics there is a smooth function f and a smooth vector field W such that $\nabla_Z Z = fZ + zW$.

Let k_λ denote the second fundamental form of the hypersurface H with respect to the metric g_λ . Let P and Q be any vectors in the plane spanned by X and Y .

$$\begin{aligned} k_\lambda(P, Q) &= (z/u)^{1/2} [z^{-1/2} g_\lambda(\nabla_P Q, Z) + z^{1/2} g_\lambda(\nabla_P Q, D)] \\ k_\lambda(\hat{Z}, P) &= (z/u)^{1/2} [-g_\lambda(W, P) + g_\lambda(\nabla_Z P, D)] \\ k_\lambda(\hat{Z}, \hat{Z}) &= u^{-1/2} g_\lambda(W, U) \end{aligned}$$

The curvature of h_λ can be expressed in terms of the curvature of g_λ and k_λ using the Gauss equation. Many of the components of k_λ appear to blow up as $\lambda \rightarrow 0$. We will see that in the case of a compact null hypersurface several apparently divergent terms vanish. **Lemma 1** Under the hypothesis of Theorem 2 the family of metrics h_λ has bounded curvature.

Proof Note first that since the dimension is three, in order to bound the sectional curvature of h_λ it suffices to show the boundedness of the components of its Ricci tensor p in an orthonormal frame. The Gauss equation gives:

$$\begin{aligned} p(P, Q) &= r(P, Q) - (\text{tr} k)k(P, Q) + (k \cdot k)(P, Q) + u^{-1} g(R(U, Q)U, P) \\ p(\hat{Z}, P) &= z^{-1/2} r(Z, P) - (\text{tr} k)k(\hat{Z}, P) + (k \cdot k)(\hat{Z}, P) + u^{-1} z^{-1/2} g(R(U, Z)U, P) \\ p(\hat{Z}, \hat{Z}) &= z^{-1} r(Z, Z) - (\text{tr} k)k(\hat{Z}, \hat{Z}) + (k \cdot k)(\hat{Z}, \hat{Z}) + u^{-1} z^{-1} g(R(U, Z)U, Z) \end{aligned}$$

The aim is to show that the quantities $p(P, Q)$, $p(\hat{Z}, P)$ and $p(\hat{Z}, \hat{Z})$ are bounded as $\lambda \rightarrow 0$. Note first that under the hypotheses of Theorem 2 the contributions of r in the above expressions are bounded. Consider next the contributions of the spacetime curvature tensor.

$$\begin{aligned} g(R(U, Z)U, P) &= -zg(R(U, D)U, P) \\ g(R(U, Z)U, Z) &= z^2 g(R(U, D)U, D) \end{aligned}$$

Thus all contributions of the curvature tensor will be bounded if we can show that $z^{-1}g(R(U, \cdot)U, \cdot)$ gives something bounded when evaluated on any pair of regular vectors. To do this it is necessary to study the behaviour of the null geodesics which are tangent to Z for $\lambda = 0$. For this computation the vector Z will be chosen to be parallelly transported along the null geodesics lying in H for $\lambda = 0$. The expansion corresponding to Z is given by $\theta = g(\nabla_X Z, X) + g(\nabla_Y Z, Y)$. Define a tensor B by $B(F, G) = g(\nabla_F Z, G)$. Then, for $\lambda = 0$, the trace of B is equal to θ and $B(F, G) = B(G, F)$ whenever F and G are tangent to H . The evolution of θ is given by the Raychaudhuri equation ([20], p. 222):

$$\nabla_Z \theta = -[B(X, X)]^2 - [B(Y, Y)]^2 - 2[B(X, Y)]^2 - r(Z, Z)$$

By hypothesis $\theta = 0$ and $r(Z, Z) = 0$. It follows that $B(X, X) = B(X, Y) = B(Y, Y) = 0$. Combining this with the condition that Z is parallelly transported shows that $B(F, G) = 0$ for any F, G tangent to H . Now suppose that F and G are vectors tangent to H which are parallelly transported along the integral curves of Z . The evolution of $B(F, G)$ is described by the equation

$$\nabla_Z(B(F, G)) = -B(F, X)B(G, X) - B(F, Y)B(G, Y) + g(R(Z, F)Z, G)$$

It follows that $g(R(Z, F)Z, G) = 0$ and hence $g(R(U, F)U, G)$ vanishes for $\lambda = 0$. This bounds the contributions of the curvature tensor in the expressions for $p(F, G)$. It remains to consider the contributions of the second fundamental form. Note that the discussion of the Raychaudhuri equation allows us to conclude and that $k(P, Q)$ is $O(z^{1/2})$ as $\lambda \rightarrow 0$. Also $\text{tr} k$ is $O(z^{-1/2})$. It follows immediately that $p(P, Q)$ is bounded. To see that $p(\hat{Z}, P)$ is bounded it suffices to observe that both $(\text{tr} k)k(\hat{Z}, P)$ and $(k \cdot k)(\hat{Z}, P)$ differ from $k(\hat{Z}, \hat{Z})k(\hat{Z}, P)$ by bounded quantities so that the singular terms cancel. Finally,

$$\begin{aligned} & -(\text{tr} k)k(\hat{Z}, \hat{Z}) + (k \cdot k)(\hat{Z}, \hat{Z}) \\ & = (-k(\hat{Z}, \hat{Z}) - k(X, X) - k(Y, Y))k(\hat{Z}, \hat{Z}) + (k(X, \hat{Z}))^2 + (k(Y, \hat{Z}))^2 + (k(\hat{Z}, \hat{Z}))^2 \\ & = -(k(X, X) + k(Y, Y))k(\hat{Z}, \hat{Z}) + (k(X, \hat{Z}))^2 + (k(Y, \hat{Z}))^2 \end{aligned}$$

so that $p(\hat{Z}, \hat{Z})$ is also bounded.

Proof of Theorem 2 If a family of Riemannian metrics is such that the curvature is bounded and the volume goes to zero then it follows from Bishop's theorem [2] that the injectivity radius goes uniformly to zero. Hence it can be concluded from Lemma 1 that the family of metrics h_λ associated to a divergence-free compact null hypersurface as above does represent a collapse in the sense of Cheeger and Gromov and that the manifold H collapses.

3. Compact Cauchy horizons

In this section theorem 2 will be applied to general relativity. A Lorentz manifold (M, g) satisfies the Einstein equations if $r = T - \frac{1}{2}\text{tr} T g$, where T is the energy-momentum tensor. What exactly this energy-momentum tensor is depends on the assumed matter

content of spacetime. The tensor T is said to satisfy the weak energy condition if $T(V, V) \geq 0$ for any timelike vector V . The following result can be deduced from theorem 2. For definitions of the concepts used in its statement, see [11].

Theorem 3 Let (M, g) be a solution of the Einstein equations with energy-momentum tensor T containing a partial Cauchy surface with a smooth compact Cauchy horizon H . Suppose that:

- (i) T satisfies the weak energy condition
- (ii) if $T(N, N) = 0$ for a null vector N then $T(N, X) = 0$ for any vector X orthogonal to N

Then H collapses with bounded diameter.

Proof Since H is a smooth compact Cauchy horizon it has zero divergence ([11], pp. 295-298). Applying the Raychaudhuri equation and using condition (i) above shows that $r(Z, Z) = 0$. Hence, by condition (ii) $r(Z, X) = 0$ for any X . Thus the hypotheses of theorem 2 are satisfied.

In order that this theorem be interesting it is necessary to check that it is satisfied for types of energy-momentum tensor which are important in general relativity. Some examples will now be given.

Example 1 $T = 0$. This is the case of the vacuum Einstein equations.

Example 2 $T = (\rho + p)U \otimes U + pg$, where U is a unit timelike vector and ρ and p are positive functions with $p < \rho$. This is the case of a perfect fluid where the speed of sound is less than the speed of light.

Example 3 T is the energy-momentum tensor of a kinetic matter model. See [15] for further discussion of this and the other matter models above.

Example 4 $T = F \cdot F - \frac{1}{4}(|F|^2)g$, where F is a two-form. This is the energy-momentum tensor of an electromagnetic field.

Theorems 1 and 2 give necessary conditions for a compact manifold to occur as a compact Cauchy horizon. This will now be complemented by mentioning examples of manifolds which do occur as Cauchy horizons in solutions of the vacuum Einstein equations. It fact it follows from [5] that manifolds occur which admit a geometric structure belonging to any Thurston geometry except Sol and hyperbolic. Since hyperbolic manifolds have been ruled out, the one geometry for which neither positive or negative results are known is Sol. Manifolds with a geometric structure of type Sol can collapse [6]. What is not clear is whether this collapse can be realized by a Cauchy horizon in a solution of the Einstein equations. This issue is related to a conjecture of Isenberg and Moncrief [12]. This says that any analytic vacuum spacetime containing a compact Cauchy horizon must admit at least one Killing vector with closed orbits which is tangent to the horizon (possibly under some additional hypotheses). This means in particular that H admits a foliation by circles, i.e. that it is a Seifert manifold [18]. This in turn means that it admits a geometric structure which is not Sol or hyperbolic. In other words, if the Isenberg-Moncrief conjecture is correct then a manifold with a geometric structure of type Sol cannot occur as a compact Cauchy horizon (at least under the assumptions of analyticity and the vacuum Einstein equations). Thus Theorems 1 and 2 would not capture all restrictions on the topology of a compact Cauchy horizon. On the other hand, pushing further the idea of applying collapsing to Cauchy horizons might be helpful in proving the conjecture, which has as yet

only been proved in a special case.

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